### Summary of Equations and Cash Flow Diagrams

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<td><strong>Present worth, P, of an arithmetic gradient series, G</strong></td>
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Engineeig Economics - Dr. Woods
Present worth, $P$, of a geometric gradient series
($n =$ number of end of period payments)

1. When $r > 1$, then $w = \frac{1 + r}{1 + i} - 1$, and
   \[ P = \frac{C}{(1 + i)^n} \left( F/A, w, n \right) \]
   \[ = \frac{C}{(1 + i)^n} \left( \frac{(1 + w)^n - 1}{w} \right) \]

2. When $r < 1$, then $w = \frac{1 + r}{1 + i} - 1$, and
   \[ P = \frac{C}{(1 + i)^n} \left( P/A, w, n \right) \]
   \[ = \frac{C}{(1 + i)^n} \left( \frac{(1 + w)^n - 1}{w(1 + w)^n} \right) \]

3. When $r = i$, then $P = \frac{Cn}{1 + r} = \frac{Cn}{(1 + i)}$

$F/C \rightarrow \infty$

$P/C = \frac{1}{(1 + r)^n}$

$P/C \rightarrow \infty$
Figure 3.3  Equivalent future value ($F$) from a series of uniform periodic amounts ($A$)

Figure 3.4  Incorrect representation of $F/A$ relationship, with extra (initial) deposit

Figure 3.5  Incorrect representation of $F/A$ relationship, with missing (final) deposit
APPLICATIONS FOR THE F/A AND A/F EQUATIONS

F/A problems involve the four variables, F, A, i, n. Whenever any three of the variables are known, the fourth may be determined. The following four examples illustrate typical situations involving each of the four variables, in turn, as the unknown.

EXAMPLE 3.1 (Find F, Given A, i, n)
Aiming to replace their mainframe computer five years from now, a firm of consulting engineers is depositing $10,000 per year into an investment fund earning interest compounded at 10 percent annually. Find the balance in the fund (including deposits and accumulated interest) at the end of five years.

Alternate Solution 1. Using the equation:

\[ F = A\left[\left(1 + i\right)^n - 1\right]/i \]
\[ F = $10,000\left[\left(1.10\right)^5 - 1\right]/0.10 \]
\[ F = $10,000 \times 6.1051 = $61,051 \]

Alternate Solution 2. Using the tables: For many problems a solution using the tables for F/A and A/F in Appendix A saves time. The value of \( \left[(1 + i)^n - 1]/i \) is represented by F/A, and the inverse is represented by A/F. Each table contains values for only one interest rate, i. The time periods, n, are shown in the left and right outer columns of each table.

1. Find the table in Appendix A for i = 10 percent.
2. Proceed down the n column to find the row where n = 5.
3. At the n = 5 row proceed horizontally to the F/A column finding F/A = 6.1051.
4. Since F/A = 6.1051, then \[ F = A \times 6.1051 \] Since \[ A = $10,000 \], then \[ F = $10,000 \times 6.1051 = $61,051 \]
EXAMPLE 3.2  (Find $A$, Given $F$, $i$, $n$)
A young engineer, now 22, expects to retire in 40 years at age 62. He anticipates that a lump sum retirement fund of $400,000 will see him nicely through his sunset years. How much should be deposited annually into an investment fund earning 9 percent compounded for the next 40 years to accumulate a $400,000 retirement fund?

Solution. Using the 9% table in Appendix A:

$A = F(A/F, i, n)$
$A = $400,000($A/F, 9\%, 40$)
$A = $400,000(0.00296$)
$A = $1,184/yr
EXAMPLE 3.3 (Find n, Given A, F, i)
A city finances its park acquisitions from a special tax of $0.02 a bottle, can, or glass on all beverages sold in the city. It derives $100,000 a year from this tax. The city has an option to buy a 40-acre lakefront park site for $1,000,000 as soon as sufficient funds accumulate. How soon will this be if the tax receipts are invested annually at the end of each year at 6 percent?

Alternative Solution 1. By logarithms:

\[ F/A = [(1 + i)^n - 1]/i \]

Therefore

\[ n = \frac{\ln(Fi/A) + 1}{\ln(1 + i)} \]

\[ n = \frac{\ln(1,000,000 \times 0.06 + 1)}{\ln(1 + 0.06)} = 8.07 \text{ yr} \]

The same result may be obtained by an alternative method.

Alternative Solution 2. By interpolation of the tables, Appendix A:

\[ A = F(A/F, i, n) \]

\[ 100,000 = 1,000,000(A/F, 6\%, n) \]
NEGATIVE INTEREST RATE

If the retirement community in Example 3.8 already has an existing population at PTZ, the solution involves one more equation and a slightly different logarithmic equation, as follows.

EXAMPLE 3.8A  (Find $n$, Given $F$, $P$, $A$, $i$) 
Assume the retirement community already has a base population of 10,000 persons. Now how long will it take to reach a population of 40,000 with all other factors the same as Example 3.8?

Solution. Given

$F = 40,000$ pop.
$P = 10,000$ pop.
$A = 6,000/yr$
$i = -0.006$

Find $n$.

$F_1 = P(1 + i)^n$  
for the 10,000 base population

$F_2 = A \left[ \frac{(1 + i)^n - 1}{i} \right]$  
for the 6,000 per year addition

$F = F_{total} = F_1 + F_2 = P(1 + i)^n + A \left[ \frac{(1 + i)^n - 1}{i} \right]$

$F = P(1 + i)^n + A \frac{(1 + i)^n}{i} - \frac{A}{i}$

$(1 + i)^n = \frac{F + A/i}{P + A/i}$

$\ln \left( \frac{F + A/i}{P + A/i} \right)$

$n = \frac{\ln(1 + i)}{\ln(1 + i)}$

$\ln \left( \frac{40,000 + 6,000/(-0.006)}{10,000 + 6,000/(-0.006)} \right)$

$n = \frac{\ln(1 - 0.006)}{\ln(1 - 0.006)} = 5.11 \equiv 5.1 \text{ yr}$
EXAMPLE 3.8 (Negative Interest Rate)

A new retirement community will need to add a second unit to the new water treatment plant when the population reaches 40,000. The community has just been built and is starting out now with a zero population. An average of 6,000 new residents are expected to move into the community each year. The estimated birthrate for the community is 8 per 1,000 and the death rate is 14 per 1,000 resulting in a negative annual increase of −0.6 percent. How long will it be before the second unit is needed for the new water treatment plant?

\[
\begin{align*}
F &= 40,000 \\
\Delta &= 6,000 \\
i &= -0.6\% \\
n - 1 &= n
\end{align*}
\]

Figure 3.10 Cash flow diagram for Example 3.8.

Solution The population must reach 40,000 before the second unit is needed, so that \( F = 40,000 \). The number of new residents expected each year is 6,000, so \( \Delta = 6,000 \). The residents that are there should have a natural rate of increase of −0.6 percent, so \( i = -0.006 \).

A diagram similar to the cash flow line diagram may be drawn for this problem as shown in Figure 3.10. The similarity to previous problems involving deposits into an interest bearing account should be apparent. In this case, the accumulated amount “on deposit” loses at rate \( i \) instead of gaining at rate \( i \).

To solve for the number of years, \( n \), that will yield a net population increase of 40,000, use the \( F/A \) equation, as follows

\[
n = \frac{\ln(Fi/\Delta + 1)}{\ln(1 + i)}
\]

\[
= \frac{\ln\left(\frac{40,000 \times (-0.006)}{6,000} + 1\right)}{\ln(1 - 0.0006)}
\]

\[
n = 6.78 = 6.8 \text{ yr}
\]

Note that since the negative \( i \) is small, and \( n \) is not large either, the number of years required to reach 40,000 is just a little more than if \( i \) were zero (40,000/6,000 = 6.67 = 6.7 yr).
CAPITALIZED COST OF PERPETUAL LIVES, \( n \to \infty \)

Capitalized cost has two popular meanings. Accountants use capitalized cost to
describe expenditures that may be depreciated over more than one year, as con-
trasted to expenditures that may be written off (expensed) entirely in the year
they were made. The second meaning for capitalized cost is found in the tra-
ditional literature of engineering economy. Here capitalized cost refers to the
equivalent lump sum amount required to purchase and maintain a project in
perpetuity (net present worth for \( n \to \infty \)). Capitalized cost problems may involve
one or more of the following categories of payments:

a. An infinite series of uniform payments.
b. An infinite series of gradient payments (arithmetic or geometric).
c. Lump sum first cost of purchase or construction
d. A series of lump sum costs of future replacements required for perpetual
   service or periodic major maintenance (the interval between payments
   is some multiple of the interest compounding period).
e. Periodic gradients to replacement costs or periodic maintenance.
a. Where only an infinite series of uniform costs are involved, the capitalized cost is simply the lump sum amount required now in an interest bearing account that will produce an infinite series of interest payments sufficient to meet all uniform costs. For instance, the capitalized cost, \( P \), equals the annual cost, \( A \), divided by the interest rate, \( i \), or, the capitalized cost = \( P = A/i \). An example follows.

**EXAMPLE 7.4 (Find the Capitalized Cost)**

The annual cost of maintaining a certain right-of-way is $10,000. Find the capitalized cost with \( i = 8 \) percent.

**Solution.** Finding the capitalized cost means find the PW with \( n \to \infty \).

\[
\text{capitalized cost} = \frac{10,000}{0.08} = \$125,000
\]

*Mental picture:* If funds could be invested at 8 percent, how much should be invested so that the interest alone would pay for the right-of-way maintenance? (Ans. $125,000)

Note that where lives are comparatively long, the capitalized cost closely approximates the PW. For example, if the time in the example above had been 100 years instead of infinity, then the PW would only be reduced to

\[
\text{PW} = \frac{10,000}{P/A, 8\%, 100} = \$124,940
\]

compared to the $125,000 found for an \( n \to \infty \).

b. Where an arithmetic gradient is involved simply apply the equation for PW of an arithmetic gradient for \( n \to \infty \), \( P = G/i^2 \). For instance, in Example 7.4, if the maintenance cost increases by $1,000 per year per year, the capitalized cost increases by

\[
P = \frac{1000}{0.08^2} = \$156,250
\]

If instead of an arithmetic gradient, a geometric gradient occurs, then apply the \( P/C \) equation (providing that \( r < i \)), in which case

\[
\text{capitalized cost, } P = \frac{C}{(1 + r)^n}
\]
For example, if the maintenance costs increase by 5 percent per year then

capitalized cost \( P = \frac{10,000}{1.05 \times 0.02857} = \$333,333 \)

c. Where a first cost is involved in addition to the PW of uniform and gradient series payments, simply add the sums to obtain the total capitalized cost of the project. For instance, in Example 7.4, if the first cost (cost of acquisition) of the right-of-way were \$47,000 then the total capitalized cost of the project would be the sum of the following costs:

\$125,000 (for maintenance), plus
\$156,250 (for arithmetic gradient maintenance), plus
\$ 47,000 (for first cost)

\$328,250 capitalized cost (NPW)

d. Where a facility is needed for perpetual service, but needs replacement from time to time, additional annual interest income must be available to accumulate to the replacement cost at the appropriate time. For instance, in the above example assume drainage structures in the right-of-way have an estimated life of 20 years and cost \$120,000 to construct now, and \$100,000 to replace at the end of every 20-year period. The cost of \$120,000 to construct now is simply added to the capitalized cost as in (b) above. The \$100,000 cost to replace at the end of every 20 years is accumulated by annual payments, \( A \), deposited into a sinking fund (savings account bearing compound interest, \( i \)). The amount required for the annual deposit to accumulate to \$100,000 at the end of every 20 years is calculated as

\[ A = \$100,000 (A/F, 8\%, 20) = \$2,190 \]

The amount of capitalized cost required to generate this annual income forever at 8 percent is

\[ P = \frac{A}{i} = \frac{\$2,190}{0.08} = \$27,400 \]

e. When the replacement costs increase by either an arithmetic or geometric gradient, the equivalent capitalized cost may be calculated using the effective interest rate, \( r_e = (1 + i)^m - 1 \). For instance, assume the cost of the replacement drainage structure previously described in (d) doubles every 20 years \( (r_{20} = 100\%) \). Then the effective interest rate for compounding once every 20 years (instead of every year at 8 percent) is found as

\[ r_e = 1.08^{20} - 1 = 3.661 \text{ or } 366.1\% \]

Since \( r_e < r_e \), then

\[ P = C(P/A, w, r)/(1 + r) \]

where \( w = (1 + r)/(1 + r) - 1 = 4.661/2 - 1 = 1.330 \)

\[ P = \$100,000/(1.330 \times 2) = \$37,580 \]
Thus, the total capitalized cost of the project for parts (a) through (e) is:

- (a) annual maintenance $125,000
- (b) arithmetic gradient maintenance $156,250
- (c) first cost R/W $47,000
- (d) first cost drainage $120,000
- (e) geometric gradient 20-yr replacement of drainage $37,580

Total capitalized cost $485,830

This is the total lump sum amount required now in an account bearing 8 percent interest in order to pay all the anticipated costs of the project for an infinite period of time.

Unfortunately, the term capitalized cost when applied to engineering economy sometimes gives rise to confusion since it is more frequently encountered in the accounting sense. The engineering use of the term probably could be laid aside and neglected by engineering students were it not for its frequent appearance on licensing exams.
GIVEN THE ANNUAL SERIES, FIND THE GRADIENT

Many practical situations occur where the present level of income is not adequate to cover the payments required to amortize the capital cost. Whenever the present worth is greater or less than the equivalent annual series a gradient may be found to make up the difference. This type problem is illustrated in the following example.

EXAMPLE 8.8  (Given $P, A, G$, Periodic Lump Sum Payments, $F, i, n$; Find NAW, $G$)

A 100-unit university married housing project (that uses state bond funds at about 6 percent) is expected to incur the following costs and income:

\[
\begin{align*}
\text{cost new} &= $1,000,000 \\
\text{maintenance} &= $100,000/\text{yr} \text{ increasing } $2,000/\text{yr} \\
\text{major overhaul every 10 yr} &= $200,000 \\
\text{salvage value at the end of 40 yr} &= $100,000
\end{align*}
\]

a. What annual income is required for this project to break even?
b. The initial annual rental income is $1,500 per unit per year ($150,000/yr for the entire project). An annual increase in rent is contemplated, since this initial income level is not sufficient to cover all costs. What annual arithmetic gradient increase will be necessary every year so that income over the 40-year life will be sufficient to cover all costs (provided the books need not be balanced until the end of the 40 years).

**Solution**

\[
A_1 = $1,000,000(A/P, 6\%, 40) = -$66,500
\]
maintenance
$A_2 = 100,000$ = $-100,000$
maintenance gradient
$A_3 = 2,000(A/G, 6\%, 40) = -$ 24,720
major renovation balance in the renovation account at EOY 40
$A_4 = 200,000(A/F, 6\%, 10) = -$ 15,180
plus salvage
$A_5 = (200,000 + 100,000)(A/F, 6\%, 40) = +$ 1,950
net annual worth = $-204,450$
a. The annual income required for breakeven is $-204,450$.
b. The required gradient is found as:

$-204,450$ annual income required
+ 150,000 actual income
- 54,450 deficit per year
required gradient $G = 54,450(G/A, 6\%, 40) = $4,406
1/12.359

The project could be supported with an initial annual income of $150,000 in the first year and an increase of $4,406 each year thereafter.
Most construction projects are built with borrowed money. Contractors usually get paid after completion of work, either:

1. Periodically (e.g., monthly) according to the amount of work completed in each month
2. By stages of completion (e.g., 15% on completion of foundation, 30% more on completion of framing, etc.).
3. Payment in full only upon satisfactory completion of the entire job.

In order to pay for labor and material before receiving payment from the owner, contractors normally are able to borrow money from a bank. The bank lends the money on the good faith expectation that the job will be completed in a timely and workmanlike manner and that the owner will pay the contractor who will then repay with interest the loan at the bank. The owner on his part is likely borrowing from a mortgage lender whose loan is secured by either a construction loan mortgage during construction, or a permanent mortgage after the work is completed and accepted. In all of these cases interest is paid on the amount borrowed, and constitutes a significant cost to the project. To determine the amount of this financing cost alone requires a two-step process:

**Step 1.** Find the future worth of the accumulated principal plus interest.
**Step 2.** Subtract the amount of principal borrowed, then the amount remaining is the total amount of interest paid in terms of dollars.

The procedure is illustrated in the following examples.
EXAMPLE 9.6 (Given $A, G, i, n$: Find Dollar Amount of Interest Cost)

A contractor is bidding on a construction project on which the owner requires completion of the work by EOM 12. The owner will pay the contractor in one lump sum at EOM 14. To pay for the day-to-day labor and material costs on the project, the contractor will require a loan from the bank of the amounts shown below. The bank charges interest at $i = 1\%$ per month on the outstanding balance.

<table>
<thead>
<tr>
<th>EOM</th>
<th>Amount borrowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 through 12</td>
<td>$100,000/month</td>
</tr>
</tbody>
</table>

Thus at the EOM 12 the contractor will have borrowed from the bank 12 payments of $100,000 each for a total of $1,200,000 and will owe the bank the $1,700,000 plus accumulated interest. The owner is scheduled to pay the contractor at EOM 14, at which time the contractor will repay the bank. How much should the contractor include in the bid just to cover the cost of financing the job?

Solution. The amount of interest can be found by finding the future worth of interest plus principal, and then subtracting the amount of the principal borrowed.

\[
F\text{W} @ \text{EOM 14} = 1,000,000(F/A, 1\%, 12)(F/P, 1\%, 2) = 1,293,700
\]

\[
\begin{align*}
\text{amount borrowed} &= 1,200,000 \\
\text{amount of interest only} &= \$93,700
\end{align*}
\]

Conclusion: In order to account for the interest costs, the contractor will have to add to his bid a financing cost of $93,700.

Financing Costs Where Monthly Progress Payments to the Contractor Do NOT Equal the Paid out Costs of the Contractor

Commonly the owner does not require the contractor to carry all the costs of financing the project during construction, but will make monthly (or other periodic) payments to the contractor. However, there is usually a retainage by the owner of a portion of the payment in order to guarantee completion, so the contractor still needs to borrow some funds in order to construct the project. To calculate the amount of the financing cost borne by the contractor, the same basic approach is employed. First, find the future worth of accumulated principal plus interest, then subtract the principal portion. The following example illustrates the process.
**EXAMPLE 9.7** (Given A, G, i, n; Find the Net Amount of Interest Cost (in Dollars) to the Contractor)

A contractor is bidding on a construction project with the anticipated cash flows shown. He borrows and lends at 1 percent per month.

<table>
<thead>
<tr>
<th>EOM</th>
<th>Income to contractor from owner</th>
<th>Cost to contractor paid out for labor and materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0</td>
<td>$100,000</td>
</tr>
<tr>
<td>1</td>
<td>80,000</td>
<td>110,000</td>
</tr>
<tr>
<td>2</td>
<td>90,000</td>
<td>120,000</td>
</tr>
<tr>
<td>3</td>
<td>100,000</td>
<td>130,000</td>
</tr>
<tr>
<td>4</td>
<td>Gradient</td>
<td>140,000</td>
</tr>
<tr>
<td></td>
<td>+10,000/month</td>
<td>level</td>
</tr>
<tr>
<td>11</td>
<td>180,000</td>
<td>140,000</td>
</tr>
<tr>
<td>12</td>
<td>190,000</td>
<td>140,000</td>
</tr>
<tr>
<td>Total</td>
<td>$1,620,000</td>
<td>$1,720,000</td>
</tr>
</tbody>
</table>

To arrive at the total contract bid price, in addition to the direct labor and materials costs of $1,720,000 listed above, this contractor expects to add:

a. The cost of financing.

b. 30 percent of all costs to cover supervision, overhead, and profit.

At EOM 12, the contractor is entitled to receive whatever balance is due from the owner on the contract, in addition to the owner’s payments of $1,620,000 listed above. Find the contractor’s bid amount which consists of:

a. Direct labor and materials $1,720,000
b. Contractor’s financing costs Find

c. 30 percent markup for supervision, OH & P Sum above x 1.3

**Solution.** First, find the contractor’s financing costs for this project. Following the approach suggested above:

*Step 1.* Find the amount of interest income the contractor could earn if the payments received from the owner were deposited in an account earning 1 percent interest per month.

*Step 2.* Find how much the contractor pays for interest costs for the amount the contractor borrows.

*Step 3.* Subtract the contractor’s interest income from the interest costs.

**Future worth of Income:**

\[
F_1 = \frac{A}{1 + \frac{i}{n}} \times (F/A, \text{ 1\%}, 12) = \frac{A}{1.01} \times 1.014610 = 1.014610 \times 960,000 = \frac{1.014610}{960,000} = 960,000
\]
\[ F_2 = \$10,000 \left( \frac{F}{G}, 1\%, 12 \right) = \frac{682,510}{68.251} \]

\[ \text{total FW} = \$1,697,120 \]

\[ \text{subtract amount of income earned} = \$1,620,000 \]

Future worth of costs:

\[ \text{amount borrowed} = FW @ 1\% \]

\[ F_3 = \$100,000 \left( \frac{F}{A}, 1\%, 13 \right) = -\$1,380,900 \]

\[ F_4 = 10,000 \left( \frac{F}{G}, 1\%, 5 \right) \left( \frac{F}{P}, 1\%, 8 \right) = -109,360 \]

\[ F_5 = 40,000 \left( \frac{F}{A}, 1\%, 8 \right) = 331,420 \]

\[ \text{total FW} = -\$1,821,680 \]

\[ \text{subtract amount borrowed} = -\$1,720,000 \]

\[ \text{cost of interest on borrowed funds} = -\$101,680 \]

Next, the net financing cost to the contractor is found as the cost of interest earned.

\[ \text{cost of interest} = -\$101,680 \]

\[ \text{interest earned} = +\$77,120 \]

\[ \text{net financing cost to contract} = -\$24,560 \]

**Conclusion** The contractor should add $24,560 to the bid to cover the net costs of the contractor's share of financing the job. The contractor's bid then consists of:

- a. Direct labor and materials $1,720,000
- b. Contractor's financing cost $24,560
- c. Markup $1,744,560
- Subtotal $2,267,930
- Contractor's bid price $2,267,930
- Sum of owner's monthly payments $1,620,000
- Final payment due contractor @ EOM 12 $647,930

**Comment** In actual practice the cash flows often do not follow uniform or gradient series. In these cases the cost of financing can be determined using the same approach but treating each payment as a lump sum if necessary, finding the equivalent FW, and following the solution process above to obtain a correct result.

**PROBLEMS OF GROWTH**

Some of our greatest problems as well as our greatest opportunities revolve around population growth. As the population grows we need more of almost everything.
As population grows the whole infrastructure must grow with it, or bottlenecks, congestion, and shortages will appear. Need for the various elements of the infrastructure is usually dictated by two factors: (a) the need or consumption per person, and (b) the number of people. For instance, electrical energy consumption is generally increasing several percent per year per person simply because we are all buying more electric and electronic equipment. In areas where there are growing numbers of people, the demand for electric energy is growing due to increasing consumption per person as well as increasing number of persons. To predict the future needs for all sorts of infrastructure elements, including roads, water plants, apartment houses, schools, and electric power generators the following approach may be employed.

**EXAMPLE 9.8**  (Given $P, A, G, F_1, i$; Find $n, F_2$)

As consultant to a municipal electric system you are asked to determine what size the next new electric power generator should be. You are given the following criteria and data.

- present number of customers on the system = 8,000
- expected new customers added to system = 400/yr
- average peak demand per customer = 4 kW
- expected increase in peak demand per customer = 3%/yr
- capacity of existing generator = 40,000 kW

The consumption characteristics for the new customers are expected to be the same as for the existing.

a. When will the capacity of the existing generator be reached by the increasing peak load?
b. Assume the new generator should be sized so that together with the existing generator they should be adequate to handle the peak load for a period of 10 years beyond the startup date of the new generator. What size should the new generator be in terms of kilowatts?

**Solution**

a. $F_1$ represents the peak demand needs of the existing 8,000 customers, which is growing by 3 percent per year, and is like a savings account with no new deposits but just earning 3 percent interest (increase) compounding each year.

$$F_1 = 4 \text{ kW} \times 8,000(F/P, 3\%, n)$$

$b. F_2$ represents the peak demands of the 400 new customers per year, and is like a series of deposits into a savings account, with the rising new balance earning 3 percent interest (increase) each year.

$$F_2 = 4 \text{ kW} \times 400(F/A, 3\%, n)$$

The sum of $F_1 + F_2$ represents the total demand for electrical energy
When the demand of these two groups totals 40,000 W, a new generator needs to go on line.

\[ F_1 + F_2 = 40,000 \text{ kW} \]

The appropriate values are substituted into the equations and they can be solved directly for \( n \) as shown below.

\[ 32K(1.03^n) + 1.6K(1.03^n - 1)/0.03 = 40K \]
\[ n = \ln \frac{1.0938}{\ln 1.03} = 3.03 \text{ yr until the capacity of the existing generator is reached} \]

b. The second question requires us to find the peak demand at a time 10 years past the time when the existing generator reaches capacity, or at EOY \( n = \text{EOY}(10 + 3.03) = \text{EOY } 13.03. \)

\[ F_1 = 32K(1.03)^{13.03} = 47.035 \]
\[ F_2 = 1.6K(1.03^{13.03} - 1)/0.03 = 25.058 \]
\[ F_1 + F_2 = \text{(capacity required @ EOY 13.03)} = 72.093 \]

less existing capacity = 40,000

new capacity required until EOY 13.03 = 32,093 MW

A similar approach may be employed to solve a wide range of problems involving planning for other future needs.

**Forecasting Growth**

Since growth occurs in many areas that affect both our personal as well as professional lives, growth normally must be taken into account when planning for the future. Admittedly, forecasts very far into the future typically are difficult to do with any great degree of accuracy. However, it is usually far more profitable to work with some estimate of future growth no matter how inaccurate than to try to ignore growth entirely. Thus some values for growth can be derived from past history, and from examining the history of other situations similar to the one under study. Projection of past history can be hazardous if not modified by the changing pressures of current and future influences. In addition, of course, the best projections of experts on growth in the area under study should be consulted. Where future projections are required it is often prudent to project a range of values, including:

1. A high "optimistic" projection, but perhaps with a common sense caveat that there is still a 10 percent (or other appropriate number) probability that even this high projection may be exceeded.
2. An expected projection, if everything goes according to the best predictions.
3. A low "pessimistic" projection with a 10 percent probability that even that projection will not be met.
FUTURE WORTH METHOD OF COMPARING ALTERNATIVES

Using these estimates of growth will enable us to better plan and prepare for whatever growth does occur. It has almost always proven more profitable to make some estimate rather than no estimate at all.

COMPARING ALTERNATIVES

As with present worth and annual worth, comparisons between alternatives may be made using future worth equivalents. Two types of alternatives commonly occur: those involving costs only and those involving incomes (or benefits) as well as costs. For problems involving costs only, the alternative with the lowest equivalent future worth (cost) is usually preferable. For alternatives involving incomes as well as cost the alternative with the greatest net future worth (NFW) is usually better, with due consideration being given the "do nothing" alternative. The following example illustrates the method.

EXAMPLE 9.9  (Given Competing Alternatives with $P, A, i, n_1 \neq n_2$; Find and Compare the NFW)

A construction company needs to buy a new front end loader, and is trying to select one from the two whose data are listed below:

a. If $i = 8$ percent, which loader, if any, should be selected?

b. If $i = 20$ percent, which loader, if any, should be selected?

<table>
<thead>
<tr>
<th></th>
<th>Front end loader A</th>
<th>Front end loader B</th>
</tr>
</thead>
<tbody>
<tr>
<td>First cost ($)</td>
<td>-$110,000</td>
<td>-$40,000</td>
</tr>
<tr>
<td>Annual net income ($)</td>
<td>+16,000</td>
<td>+13,000</td>
</tr>
<tr>
<td>Useful life (yr)</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Replacement cost escalation</td>
<td>NA</td>
<td>10%/yr compounded</td>
</tr>
<tr>
<td>Salvage value ($)</td>
<td>+10,000</td>
<td>+10,000</td>
</tr>
</tbody>
</table>

Solution

a. For an $i$ of 8 percent,

\[
\text{NFW}(A) = -110,000(F/P, 8\%, 10) + 16,000(F/A, 8\%, 10) + 10,000
\]

\[
= \frac{-110,000 \times 2.1589 + 16,000 \times 14.4866 + 10,000}{1.08^{10}} = +$4,307
\]

In order to compare loader B which only has a five-year life span, a second loader B is purchased at EOV 5 so that the two loaders both have a 10-year service life span.

\[
\text{NFW}(B) = -40,000(F/P, 8\%, 10) - 40,000(F/P, 10\%, 5)(F/P, 8\%, 5)
\]

\[
= \frac{-40,000 \times 2.1589 - 40,000 \times 1.6106 \times 1.4693}{1.08^{10}} = +$22,005
\]
COMPARING ALTERNATIVE PROPOSALS

**Solution a.** With \( i = 8 \text{ percent} \), **Loader B** with the greatest positive NFW, is preferred.

b. For \( i = 20 \text{ percent} \),

\[
\text{NFW (A)} = -110,000(F/P, 20\%, 10) + 16,000(F/A, 20\%, 10) + 10,000
\]
\[
6.1917 \quad 25.958
\]
\[
= -\$255,759
\]

\[
\text{NFW (B)} = -40,000(F/P, 20\%, 10) - 40,000(F/P, 10\%, 5)(F/P, 20\%, 5)
\]
\[
6.1917 \quad 1.6106 \quad 2.4883
\]
\[
+13,000(F/A, 20\%, 10) + 10,000 + 10,000(F/P, 20\%, 5)
\]
\[
= -\$35,637
\]

**Solution b.** In part (b), using an \( i = 20 \text{ percent} \) both alternatives yield a negative NFW. This result indicates that both investments will yield a return of less than 20 percent. If the "do nothing" alternative is available, then no investment should be made if a 20 percent return is required.

THE "DO NOTHING" ALTERNATIVE

When considering an investment involving income, frequently the investor has the option of either investing in one of several alternatives or not investing at all (known as the "do nothing" alternative). However, in some instances the "do nothing" alternative is not available even when income is expected. The investment under consideration may be a part of a larger operation that would not function properly without this investment as an integral part. For instance, a construction contractor may spend relatively large sums on copy machines and word processors. This equipment may not make any profit on its own but could contribute significantly to the profitability of the contractor's construction firm. Similarly a department store may operate a lunch counter or a branch post office at a loss in order to attract more customers into the store. Thus, even when income is involved the "do nothing" alternative may or may not be available depending upon the particular proposal under consideration.

In part (2) of Example 9.9, the "do nothing" alternative is the better selection (if it is available) since the company can make more money by investing the purchase price of either loader in an investment earning 20 percent [if the company is borrowing at \( i = 20\% \), then just repaying some of their debts represents an investment that saves (yields) 20% rate of return] rather than in investing in either piece of equipment.

This leads naturally to the question of what rate of return will these two possible alternatives yield the company? Obviously they both will yield greater than 8 percent since in part (1) the NFW's are positive at 8 percent, and, they both yield less than 20 percent since in part (2) the NFW's are negative at 20 percent. The methodology of finding the actual rate of return for each alternative is presented in the next chapter.
SUMMARY

In this chapter, the methods for finding and comparing future worths of a variety of cash flow combinations are presented. These methods are similar in many respects to the methods used for finding present worths and annual worths. Future worth is useful for finding the future value of an investment or for comparing alternatives on the basis of net future worth at some specified rate of return. As with present worth, the lives of various alternatives must be equal in order for the comparison to be valid.
CAPITALIZED COSTS FOR INFINITE LIVES

\( P/A \to 1/i \text{ as } n \to \infty \)

Occasionally problem situations occur involving practically infinite values of \( n \).
For instance a government department of transportation (DOT) may need land

for a right of way (R/W). The land will require maintenance (mowing, etc.) for
as long as the DOT owns it, and the DOT intends to own the land for a very
long time, or perpetually for all practical purposes. If the annual cost of mainte-
nance is valued at \( A \), then the present worth of that annual cost charged at the
end of every year forever, from Equation 4.1, is

\[
P = A \left[ \frac{(1 + i)^\infty - 1}{i(1 + i)^\infty} \right] = A \left[ \frac{(1 + i)^\infty}{i(1 + i)^\infty} - \frac{1}{i(1 + i)^\infty} \right]
\]

\[
P = A \left( \frac{1}{i} \right) = \frac{A}{i} \quad \text{or} \quad A = Pi \quad \text{for} \quad n \to \infty
\]
EXAMPLE 4.7 (Given \( P_1, A_1, i, n \to \infty \): Find \( P_2, A_2 \))

A state DOT requests you to calculate the present value of acquiring and maintaining a new R/W which costs $300,000 to purchase and $7,200 per year to maintain. They intend to hold the R/W in perpetuity and ask for equivalent costs both in terms of (1) present worth (\( P \)), and (2) annual worth (\( A \)). Use \( i = 6 \) percent.

**Solution 1**  The equivalent total present cost in terms of present worth, \( P \), is

\[
\text{cost of acquisition, } P_1 = 300,000 \\
\text{cost of maintenance, } P_2 = A_i = \frac{7,200}{0.06} = 120,000 \\
\text{total equivalent present worth, } P_t = 420,000
\]

(\( P_t \) is sometimes referred to as capitalized value or capitalized cost.)

**Comment:** The $120,000 equivalent present worth of maintenance may be visualized as the amount of money needed in a fund drawing 6 percent interest so that the interest alone is sufficient to pay all of the annual costs.

**Solution 2.** The equivalent annual cost is

\[
\text{cost of acquisition, } A_1 = P_i = 300,000 \times 0.06 = 18,000/\text{yr} \\
\text{cost of maintenance, } A_2 = \frac{7,200}{25,200/\text{yr}}
\]

**Comment:** The $18,000 per year equivalent annual cost of acquisition may be viewed as the cost of annual interest if the $300,000 is borrowed and not repaid. In other words, the $300,000 could be borrowed, exchanged for the land, and then repaid whenever the land is resold at the same purchase price. The only cost of ownership would be the interest cost on the borrowed money.
PERPETUAL LIVES

Some investments, such as land, typically do not depreciate, but appreciate in value. For investment study purposes the land underlying a project is often viewed (conservatively) as merely maintaining its original value. The annual cost of such a land investment may be pictured as the annual amount required to pay interest on a loan used to acquire the land. If at some future time the land is sold for the same amount as the purchase price, then the loan is paid off and the only costs have been the interest payments. (If the land or other investment is expected to sell at an increased price the increment of increase can be separated from the purchase price and the $A/F$ equation used to find the equivalent annual income payment.)

EXAMPLE 8.6  \( \text{Given } P, i, n = \infty; \text{ Find } A/W \)

Some right-of-way land is needed for a highway that will be in use for a very long time \( (n \to \infty) \). If the land costs $100,000, and \( i = 8 \) percent, find the equivalent annual cost of the land.

Solution

\[ A_1 = IP = 0.08 \times 100,000 = 8,000/\text{yr} \]

\textit{Mental picture:} Borrow the money to buy the land. The only funds needed is the money to pay the annual interest payment of 8 percent, or $8,000 per year on the $100,000 loan. It is assumed that at any time in the future, should the need arise, the land could be sold for $100,000 and the loan repaid.
EXAMPLE 5.4  (Given A, G, i, n. Find Equivalent A)

A bridge authority collects $0.25 per vehicle and deposits the revenue once a month in an interest bearing account. Currently 600,000 vehicles a month cross the bridge and the traffic count is expected to increase at the rate of 3,500 vehicles per month added for each succeeding month. What equivalent uniform monthly revenue would be generated over the next five years if the interest in the account were (1) 6 percent nominal interest compounded monthly (0.5 percent per month), and (2) 6 percent compounded annually?

Solution for Part 1, with Monthly Compounding  The monthly costs for the problem in terms of A and G are determined as follows:

\[ A = 600,000 \text{ vehicles/month} \times $0.25/\text{vehicle} = $150,000/\text{month} \]

\[ G = 3,500 \text{ vehicles/month} \times $0.25/\text{vehicle} = $875/\text{month} \]

\[ n = 5 \text{ yr} \times 12 \text{ months/yr} = 60 \text{ months} \]

The cash flow diagram is shown in Figure 5.11.

\[ A_1 = $150,000/\text{month} = $150,000/\text{month} \]

\[ A_2 = 875(A/G, 0.5\%, 60) = $24,500/\text{month} \]

\[ A = A_1 + A_2 = $174,500/\text{month} \]
Solution for Part 2, with Annual Compounding. Since interest in part (2) is earned only once each year the payments can be grouped together at the end of each year, negating any benefit from depositing at monthly periods (see Chapter 4 for a discussion of this situation). Thus, the base annual income deposited at EOY 1.

\[
A_1 = [600,000 \text{ vehicles/month} \times 12 + 3,500 \text{ vehicles/month/month}) \times (1 + 2 + \cdots + 11)] \times 0.25/\text{vehicle} = \$1,857,750/\text{yr}
\]

The gradient income deposited at the end of each subsequent year.

\[
G = 3,500 \text{ vehicles/month/month} \times (1 + 2 + 3 + \cdots + 12) \times 0.25/\text{vehicle} = 68.750/\text{yr/yr}
\]

See Figure 5.12 for the cash flow diagram. Solving for the annual \(A\) yields

\[
A = \$1,857,750/\text{yr}
\]

\[
A_2 = 68,250(A/G, 6%, 5) = \frac{128,656}{1.8836} = \$1,986,300/\text{yr}
\]

annual \(A\) = \$1,986,300/\text{yr}

Since interest is earned only at the end of each annual period, the monthly equivalent is simply one-twelfth of the annual equivalent. Thus

\[
\text{monthly } A = \frac{1,986,300}{12} = \$165,500/\text{month}
\]

(Compare to \(A = \$174,500/\text{month} \) in Solution 1.)

\[
\begin{array}{c}
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\hline
A = \$1,857,750/\text{yr} \\
G = 68,250/\text{yr/yr}
\end{array}
\]

\(r = 6\%\)

Figure 5.12 Cash flow diagram for Example 5.4. Solution 2.

This answer is significantly different from Solution 1, even though the interest rate does not appear to be too different. (For Solution 1 the annual \(i_r = (1.05)^{12} - 1 = 6.17\%\) versus 6% in Solution 2.) The reason for this difference is the simplifying assumption that receipts that occur between the points in time at which compounding occurs (intermediate funds flow earn no interest until the next complete compounding period ends. Thus the precise terms of interest compounding periods and earnings on intermediate funds flow become very important in actual practice.
PERPETUAL LIFE WITH GRADIENT INCREASES

Some perpetual life problems involve a gradient increase in cost. For instance, assume the maintenance costs on each mile of right-of-way increase by $100 per year over a perpetual life. To find the present worth of this arithmetic gradient, simply solve Equation 5.8, letting \( n \) approach infinity.

\[
P/G = 
\frac{1}{i}
\left[
\frac{(1 + i)^n - 1}{i(1 + i)^n} - \frac{n}{(1 + i)^n}
\right]
\]

Consider the elements of the equation one at a time. As \( n \) approaches infinity both the numerator and the denominator of both parts of the equation become very large. To analyze, use L'Hospital's rule, as \( n \) approaches infinity, \((1 + i)^n\) becomes very large and also approaches infinity.

As \( n \to \infty \)

\[
\frac{n}{(1 + i)^n} = \frac{d(1 + i)^n}{dn} = \frac{1}{n(1 + i)^{n-1}} = 0
\]

Therefore

\[
\frac{n}{(1 + i)^n} \to 0 \quad \text{as} \quad n \to \infty
\]

and

\[
\frac{(1 + i)^n - 1}{(1 + i)^n} \to 1 \quad \text{as} \quad n \to \infty
\]
so that

\[ P/G \rightarrow \frac{1}{i} \left( \frac{1}{i} - 0 \right) \quad \text{as} \quad n \rightarrow \infty \]

or, for \( n = \infty \),

\[ P/G = \frac{1}{i^2} \quad (5.9) \]

Applying this solution to the foregoing right-of-way problem (assuming \( i = 6\% \)), the present worth of the perpetually increasing maintenance cost of $100 per year is found as

\[ P = \frac{G}{i^2} = \frac{100}{0.06^2} = 27,778 \]

In other words, if a trust fund of $27,778 earns 6 percent interest, then withdrawals of $100 may be commenced at EOY 2 and increased by $100 per year each year forever.

**Perpetual Life for Other Time-Value Equations**

As \( n \rightarrow \infty \), the other time-value equations are evaluated in a manner similar to the evaluations in the preceding paragraph. The results are as follows:

- \( A/F = 0 \)
- \( A/P = i \)
- \( A/G = 1/i \)
- \( P/F = 0 \)
- \( P/A = 1/i \)
- \( P/G = 1/i^2 \)
- \( F/P = \infty \)
- \( F/A = \infty \)
- \( F/G = \infty \)
Rate of Return Method

KEY EXPRESSIONS IN THIS CHAPTER

ROR = Rate of Return is the equivalent interest rate earned or paid on the principal balance of an investment or loan.

BASICS OF ROR

The rate of return (ROR) on an investment is a concept that is already familiar to most people. A return of $10 interest per year on a deposit of $100 is easily understood to imply a rate of return of 10 percent. Therefore, those who make decisions on where to spend the money usually appreciate the ease, speed, and convenience of comparisons presented in terms of rates of return. The project with a return of 18.2 percent obviously is more desirable than an alternative project with a return of only 9.3 percent (providing the risk and other considerations are comparable). Because the results are easily understood, the rate of return method together with the companion incremental rate of return method are powerful tools for comparing alternatives and providing decision makers with the information required to obtain a better investment of the public or private dollar, and more efficient allocation of other resources of all types.

In order to obtain a positive rate of return an investment must yield a positive net income, that is, the total income cash flow must exceed the total cost cash flow. For example, an investment of $1,000 today that will yield an income of $1,100 one year from today has a rate of return of 10% (calculated as ROR, \( i = \frac{F/P}{P} - 1 = \frac{1,100}{1,000} - 1 = 0.10 = 10\% \)). If the $1,000 investment only yields $1,000 after one year then the rate of return is zero.
whereas if the income is less than $1,000, say $900 then the rate of return is a negative 10% \( i = (900/1,000) - 1 = -10\% \).

In general terms the rate of return is the interest rate, \( i \), at which the equivalent cash flow-in equals the equivalent cash flow-out: ("equivalent" here means that the time value of money has been accounted for, and future payments are appropriately discounted.) More specifically, the rate of return is the rate, \( i \), at which:

- the PW of all costs = PW of all incomes, and NPW = 0
- the AW of all costs = AW of all incomes, and NAW = 0
- the FW of all costs = FW of all incomes, and NFW = 0

If incomes are viewed as "benefits", while payments out are termed "costs" (both of which are expressed in equivalent terms), then the rate of return is the interest rate at which the sum of all benefits are equal to the sum of all costs. This condition is satisfied when any of the following occur (assume that the "benefits" and "costs" refer to the PW, or AW, or FW of each benefit and cost):

- benefits = costs
- benefits - cost = zero
- benefits/costs = one

**EXAMPLE 10.1**

A $1,000 cost results in a lump sum payment of $2,500 at the end of 10 years. What is the rate of return (ROR) on this investment?

**Solution** The solution is obtained by formulating the income and cost into equivalent amounts by PW (finding the FW or AW of both could be done with equal ease) then solving for the value of \( i \) which makes the NPW = 0.

- PW benefits = 2,500 \((P/F, i, n) = 2,500/(1 + i)^{10}\)
- PW costs = 1,000
- NPW = 2,500/(1 + i)^{10} - 1,000 = 0
- \((1 + i)^{10} = 2.5\)
- \( i = \sqrt[10]{2.5} - 1 \)
- \( i = 0.09596 \) or \( 9.60\% \)

The problem could also be solved using NFW, or NAW.

**STEP-BY-STEP PROCEDURE FOR FINDING ROR**

The foregoing simple example could be solved directly because there was only one unknown value of \( i \). More commonly the unknown value, \( i \), appears several times in several equations, and the most practical general method of solution is by successive approximation (trial and error). The following step-by-step procedure is recommended. As happens in the usual case, this procedure assumes
that the investment cost occurs before the income returns. Therefore, the graphic plot of net worth versus i value curves downward to the right, and higher values of i are associated with lower values of net worth. If the situation were the reverse and the income came before the cost (such as when a borrower borrows money and later repays) the graph would curve upward to the right.

**Step 1.** Make a guess at a trial rate of return, i.

**Step 2.** Count the costs as negative (−) and the income or savings as positive (+). Then use the trial i value to find the equivalent net worth of all costs and income. Use either present worth, annual worth, or future worth.

**Step 3.** Using the trial i value, if the income from the investment (counted as positive value) is worth more than the cost of the investment (negative value) then the equivalent net worth is positive at this trial i value. When the cost precedes the income (usual case), the graphic plot of net worth versus i value curves downward to the right. When the trial i yields a positive net worth, we can surmise that the trial i is too low and the correct i value is higher than the trial rate, i.

**Step 4.** Adjust the trial i value upward, and proceed with steps 2 and 3 again until one value of i is found that results in a positive (+) equivalent net worth, and another value of i is found with a negative (−) equivalent net worth.

**Step 5.** Interpolate between the two trial i values (one yielding a positive net worth and the other a negative), to solve for the correct value of i. Remember that the graph of NPW versus i is a curve so the two trial values of i must be reasonably close in order for the interpolation to yield accurate results.

An example involving two lump sum cash flows as well as a periodic series cash flow will serve to illustrate the procedure.

**EXAMPLE 10.2**

Assume a bond sells for $950. The bond holder will receive $60 per year interest as well as $1,000 (the face amount of this bond) at the end of 10 years. Find the rate of return.

**Solution** Find the interest rate at which the present worth of the income ($60 per year for 10 years, plus $1,000 at the end of 10 years) equals the present worth of the cost (−$950). The steps outlined above are applied as follows.

**Step 1** Make a trial selection of 7 percent for a trial rate of return. (The yield is $60 per year on an investment of $950, plus an extra $500 at the end of 10 years, so the i probably will be a little above 60/950 = 0.063 or 6.3 percent.)
RATE OF RETURN METHOD

Step 2. Find the present worth of all costs and income using \( i = 7 \) percent. The income consists of $60 per year for ten years, plus $1,000 at the end of year 10. The cost is one $950 payment due right now.

\[
\begin{align*}
\text{income } P_1 &= 60 \text{ yr}(P/A, 7\%, 10) = 60 \times 7.024 = 421.44 \\
\text{income } P_2 &= 1000(P/F, 7\%, 10) = 0.5083 \\
\text{cost } P_3 &= -950.00 \\
\text{net present worth} &= -950.00 + 421.44 + 0.5083 = -20.26
\end{align*}
\]

Step 3. This graph curves downward to the right. Since the net present worth is negative, the trial rate, 7 percent, must be too high. The present worth of income ($929.74) is less than the present worth of costs ($950.00) or at 7 percent you are offered an opportunity to pay $950 for an income worth $929.74.

Step 4. Adjust the estimate downward, say, 6 percent. Then, the net present worth of the income minus cost is

\[
\begin{align*}
P_1 &= 60 \text{ yr}(P/A, 6\%, 10) = 60 \times 7.360 = 441.60 \\
P_2 &= 1000(P/F, 6\%, 10) = 1000 \times 0.5584 = 558.40 \\
P_3 &= -950.00 \\
\text{net present worth} &= +50.00
\end{align*}
\]

Two values of \( i \) have been found, each resulting in a different sign (+ or -) thus bracketing the value of \( NPW = 0 \). An interpolation to the correct value of \( i \) (the value of \( i \) at which the NPW equals zero) is found as follows.

<table>
<thead>
<tr>
<th>( i ) value</th>
<th>NPW</th>
</tr>
</thead>
<tbody>
<tr>
<td>6%</td>
<td>+50</td>
</tr>
<tr>
<td>ROR ( i )</td>
<td>0</td>
</tr>
<tr>
<td>7%</td>
<td>-20.26</td>
</tr>
</tbody>
</table>

rate of return, \( i = 6\% + \frac{50.00}{50 + 20.26} \times 1\% = 6.7\% \)

The 6.7\% return is called the bond’s “yield to maturity” because it includes redemption income when the bond matures. (The “current yield” is $60/ $950 = 6.3\% and is frequently used in financial circles with reference to bonds whose maturity date is fairly distant. Bonds are discussed in more detail in Chapter 17.)

CONCURRENT INCOME AND COSTS

The rate of return method can be used where both costs and income occur during the same periods. To save a step, use the net payment for those periods as in the following example.
EXAMPLE 10.3

An entire fleet of earthmoving equipment may be purchased for $2,400,000. The anticipated income from the equipment is $920,000 per year with direct expenses of $420,000 per year. The market value after five years is expected to be $1,000,000. What is the rate of return?

**Solution**

1. Assume $i = 10\%$ for first trial.
2. Find present worth of all income and costs.

\[
\begin{array}{c|c|c}
\text{income} & $920,000/yr & \\
\text{expenses} & -420,000/yr & \\
\text{net annual income} & $500,000/yr & \\
\end{array}
\]

The present worth of this net income for five years is

\[P_1 = \frac{500,000}{r}(P/A, 10\%, 5) = 1,895,350 \text{ present worth of income}\]

\[3.7907\]

Market value in five years is $1,000,000. The present worth of $1,000,000 income expected five years from now is

\[P_2 = \frac{1,000,000}{r}(P/F, 10\%, 5) = 620,920 \text{ present worth of market value}\]

Total present worth of all income at 10\% is

\[P_{1+2} = 1,895,350 + 620,920 = 2,516,270 \text{ total}\]

This means that if a 10\% return is an acceptable rate of return you can afford to pay as high as $2,516,270 cash for this project. You would then receive $500,000 per year (part of which is interest and part is principal) for five years plus $1,000,000 at the end of the fifth year, and this income yields 10\% return on the balance of all funds remaining in the investment at the end of each year. Actually this fleet is being offered for only $2,400,000.

**Step 3** Conclusion: Assuming a trial $i$ of 10\%, the income (with a PW of $2,516,270 at 10\%) is worth more than the cost (with PW of $-2,400,000), resulting in a positive NPW of $(2,516,270 - 2,400,000) = +$16,270, so the trial estimate is too low.

**Step 4** Revise the trial to 12\% and repeat steps 7 through 3. The PW at 12\% is found as

\[P_3 = \frac{500,000}{r}(P/A, 12\%, 5) = 1,802,350\]

\[3.6047\]

\[P_4 = \frac{1,000,000}{r}(P/F, 12\%, 5) = 567,430\]

\[0.56743\]

\[P_{1+2} = \text{total PW of all income at 12\%} = 2,369,780\]
**Rate of Return Method**

![Diagram](image)

Figure 10.1 Interpolation to find rate of return.

**Conclusion.** Since the PW of income at 12 percent is lower ($2,369,780) than the PW of cost ($2,400,000), resulting in a negative NPW, the rate of return is lower than 12 percent. Interpolation will give a final figure as illustrated in Figure 10.1.

\[

t = 10\% + \frac{116,270}{116,270 + 30,220} \times 2\% = 11.59\%
\]

\[
\begin{align*}
10\% PW \text{ income} &= 2,516,270 \\
PW \text{ cost} &= -2,400,000 \\
+ &\quad 116,770 \\
\end{align*}
\]

\[

t = 10\% + \frac{116,270}{116,270 + 30,220} \times 2\% = 11.59\%
\]

\[
\begin{align*}
12\% PW \text{ income} &= 2,369,780 \\
PW \text{ cost} &= -2,400,000 \\
- &\quad 30,220 \\
\end{align*}
\]

\[
\begin{align*}
10\% PW \text{ income} &= \frac{116,270}{116,270 + 30,220} \times 2\% = 11.59\%
\end{align*}
\]

\[
\begin{align*}
11.6\% 
\end{align*}
\]

This problem can also be worked by using net annual worth or net future worth.

**Gradients on Income and Costs**

Gradients present no unusual problems when finding the rate of return. Simply follow the standard procedure for determining the rate of return:

1. Assume a trial interest rate.
2. Find the equivalent worth of each payment item, treating gradient items the same as usual.
3. Sum the totals.
4. By trial, error, and interpolation find the interest rate at which the sum of equivalent worths equals zero.

The next example illustrates gradients in a rate of return problem.

**Example 10.4**

A bus company is for sale for $150,000. The net income this year is $50,000 but is expected to drop $6,000 per year next year and each year thereafter.
COMPARING ALTERNATIVE PROPOSALS

At the end of 10 years the franchise will be terminated and the company assets sold for $50,000. What is the interest rate of return on this investment?

Solution. Try \( i = 15\% \).

\[
P_i = -\$150,000
\]

\[
P_2 = -G\left(P/G, 15\%, 10\right) = 6,000 \times 16.979 = -\$101,890
\]

\[
P_1 = A\left(P/A, 15\%, 10\right) = 50,000 \times 5.0187 = +\$250,950
\]

\[
P_4 = F\left(P/F, 15\%, 10\right) = 50,000 \times 0.2472 = +\$12,360
\]

\[
P_{\text{total}} = P_1 + P_2 + P_3 + P_4 = +\$11,420 \text{ for } i = 15\%.
\]

Try \( i = 20\% \).

\[
P_{\text{total}} = -\$9,648 \text{ for } i = 20\%
\]

Interpolating.

\[
i = 15\% + \frac{11,420}{11,420 + 9,648} \times 5\% = 17.7\%
\]

Periodic Gradients. ETZ or PTZ, Financing

No special techniques are required in order to find the rate of return of an investment involving periodic gradients and financing. Simply organize the problem material in an orderly fashion and proceed with the step-by-step solution. An example follows.

EXAMPLE 10.5 (Given \( P, A, G, C, i, n; \text{ Find ROR } i \))

A government agency needs more office space. They either can rent the space on the rental market or renovate an existing older office building which they already own. Find the ROR if they renovate the existing older building and count the rent saved as income over the next 40 years.

- Market value of existing older building \(-\$1,200,000\)
- Cost of rehabilitation \(-\$4,000,000\)
- Rehabilitation loan available \( @ 3\% \) to be repaid in 10 equal annual installments, with first payment due \( @ \text{EOY } 1 = -\$3,200,000\)
- Rental value \( @ \text{EOY } 1 \) (count as income) \(-\$400,000\)
- Geometric gradient in rental value \(-5\%/\text{yr}\)
- O & M costs \( @ \text{EOY } 1 = \$90,000\)
- Geometric gradient in O & M costs \(-5\%/\text{yr}\)
- Major maintenance (MM) every 5 yr, first MM \( @ \text{EOY } 3 = \$20,000/5\text{-yr period}\)
- Gradient increase in major maintenance (MM) costs \$30,000 \@ EOY 8, \$40,000 \@ EOY 13, etc.) \(-\$10,000/\text{each/}5\text{-yr}\)
- Resale value at EOY 40 \(-\$100,000\)
This yields two roots,
\[ i = 0.1 \text{ or } 10\% \text{ and } i = 0.2 \text{ or } 20\% \]
Solving by the quadratic equation yields the same results, as does solution
by trial and error.
A graph of the rate of return versus the PW of the total investment
is plotted in Figure 10.3. Inspection of the graph indicates that if the initial
expenditure were increased by $1 (to $501), there would be only one
solution, the tangent point of the parabola with the x axis at \( i = 15 \text{ percent} \).
Further, if the initial investment were raised to more than $501, there
would be no solution, since all total PW’s of cost plus income would be
negative regardless of the trial \( i \) employed.
In connection with the above discussion, the following observations
may prove helpful.
1. Only one change or no change in the sign of the cumulative cash
flow indicates there is only one solution \( i \).
2. It is theoretically possible to have any number of solutions, pro-
viding that the sign of the cumulative cash flow changes more than once.
For example,
\[
[(1 + i) - 1.05][(1 + i) - 1.10][(1 + i) - 1.15] = 0
\]
This “solution” will solve for \( i = 5 \text{ percent} \), \( i = 10 \text{ percent} \), and \( i = 15 \text{ percent} \). This “solution” may be multiplied out to find the equivalent cash
flow, as follows:
\[
+(1 + i)^3 - 3.30(1 + i)^2 + 3.63(1 + i)^1 - 1.33 = 0
\]
Any convenient multiple of these numbers will yield to the same three
interest rate solutions.
3. Multiple rates of return, intuitively, seem contrary to reason and
common experience. However, multiple rate solutions do occur due to the
implicit assumption that all the cash flow values (both positive income
and negative costs) will earn the solution interest rate (the derived rate, \( i \).
lower operating and maintenance costs, higher resale values, greater capacity, or some combination thereof. Can the car owner afford to pay $125 more in first cost for a set of tires in order to extend the expected life from 24 months to 36 months? Even though there is no direct dollar income or "return" resulting from this type investment, the incremental rate of return (IROR) method is well suited and often used to find the most economical alternative.

Instead of focusing on the income earned by every dollar of investment as in the rate of return method, the IROR method looks at the amount saved by each additional dollar of investment above the required minimum. Thus the IROR is the return on each optional incremental dollar of capital investment under review.

The problem now assumes the following form. If $1 must be invested with no measurable return expected, and an additional (incremental) $1 may be added to this investment with a positive return (savings in O & M. longer life, etc.) expected on the second $1, should the optional second $1 be invested in addition to the mandatory first $1? The following example illustrates this situation.

**Example 11.1** (Given: Must Purchase Either A or B; Find the Incremental Rate of Return on the Increment of Cost)

A car owner has worn out tires and no alternative transportation. He can buy a set of tires expected to last two years for $200, or a set of tires expected to last four years for $325 ($125 more). Which is the better alternative?

**Solution** The solution may be found through IROR. First clarify the problem situation by drawing the cash flow diagram of each alternative. Assume that over a four-year period of time two sets of the two-year life tires would be needed, and that the replacement cost for the two-year tires at EOY 2 remains at $200 and does not change. The increment of invest-

\[
\begin{align*}
A & \quad \text{\$200} \\
& \quad \text{\$325} \\
B & \quad \text{\$200} \\
B - A & \quad \text{\$125}
\end{align*}
\]

\[B - A = \text{the difference in the first cost of the alternative sets of tires, or } (\text{\$325 - \$200 = \$125}. \text{ If the extra \$125 is paid out now, then there will be no need to buy another set of tires at the EOY 2, and a savings of \$200 will occur at that point in time. Thus an investment of an incremental \$125 now will produce a savings (or return) of \$200 at EOY 2. The ROR}\]
The do nothing alternative. Note that in the foregoing Example 11.1 it was assumed that the car must be kept in operation and tires of some description must be purchased. The car owner could not afford to do nothing, therefore the "do nothing" alternative was not available. In this case if there were no negative consequences from doing nothing, then the do nothing alternative of not buying any tires at all would have been the most economical choice.

COMPARISON WITH EQUAL LIFE SPANS

The following example illustrates an IROR comparison of alternatives all having the same life span.

EXAMPLE 11.2 (Given Two Alternative Sets of P, A, F, n; Find IROR)

Assume a coal burning electric power generating plant must select one pollution control device from two suitable types available having the characteristics listed below. The do nothing alternative is not available.

<table>
<thead>
<tr>
<th>Pollution control device</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost new</td>
<td>$1,000,000</td>
<td>$1,300,000</td>
</tr>
<tr>
<td>Operating costs</td>
<td>$300,000/yr</td>
<td>$265,000/yr</td>
</tr>
<tr>
<td>Life, n</td>
<td>20 yr</td>
<td>20 yr</td>
</tr>
<tr>
<td>Salvage value (not the same)</td>
<td>$50,000</td>
<td>$60,000</td>
</tr>
</tbody>
</table>

Recommend the preferable alternative based on the incremental rate of return (IROR).

Solution

Note: Neither of these alternatives produces income for the owner except the small salvage value at EOY 20. Therefore, there is no need to determine the ROR since it is obviously some large negative number. The problem now is to compare the extra benefits available from alternative B (save $35,000 per year in operating costs plus the small increase in salvage value) to the extra $300,000 initial cost of alternative B. To compare, first find the incremental rate of return (IROR) on the extra $300,000 investment in B, and then compare that IROR with the MARR.

Comparing the two alternatives, the lowest initial investment is required by alternative A, at $1,000,000. Alternative B requires $1,300,000 capital outlay ($300,000 more than B). This raises the basic question "since a minimum of $1,000,000 must be spent on pollution control anyway, should an extra $300,000 be invested in alternative B in order to save $35,000 per year in operating costs, plus receive $10,000 more in salvage value at EOY 20?" Will the lower operating cost and higher salvage value of B more than compensate for the extra $300,000 initial cost of alternative
OVERVIEW

Benefit cost analysis is a well-rooted method of evaluating public projects. The basic measure of acceptability is a benefit to cost ratio greater than 1.0 or, equivalently, a positive net difference between benefit and cost. Incremental B/C ratios should also exceed unity for added increments of investment. B/C = 1 is a standard of minimum approval but is not an effective ranking criterion. Discounting procedures, computational practices, and capital-returning considerations are essentially the same in the private and public sectors.

The appropriate interest rate for the evaluation of public projects is a much-discussed issue. Social projects are particularly sensitive to the rate of discount applied, because most major expenditures occur early, whereas the benefits extend many periods into the future; thus, a high discount rate tends to reduce the relative proportion of benefit to cost. Low discount rates are recommended by those who hold it is government's social responsibility to undertake projects that do not necessarily provide immediate returns comparable with those earned in the private sector. Counterarguments claim that resources are wasted when transferred from private uses, where returns are high, to low-yield public investments. One measure of the social discount rate results from computing the total cost of government borrowing.

Social projects often provide public goods or services available to all if available to one. The actual amount paid for a public good may not be a true measure of its worth. A common example occurs when a good is worth more to the user than the price paid for it. Spillover benefit or cost occurs when an activity affects those parties not directly involved in a project or program. These spillovers are additional
failures in government-funded projects and proposals. Therefore, as is the case with investments in the private sector, the social discount rate should include a risk factor. A surcharge of ½ to 1 percent has been suggested, but this across-the-board addition fails to account for differences in the nature of risk for various classes of projects.

BENEFIT-COST APPLICATIONS

Many of the charges of misuse of benefit-cost analyses are the result of specific projects that have gone awry. The well-publicized cases of dams that failed to stop flooding as promised and project overruns that boosted costs well above anticipated benefits could be considered as evidence of the need for more exacting benefit-cost analysis, rather than of deficiencies in the B/C criterion.

In addition to the theoretical critiques of the applicable discount rate and the fairness of benefit measurements, criticism has been leveled at the way the analyses are conducted. It is claimed that unrealistically high values have been assigned to intangibles to compensate for low monetary benefits in projects that would be unacceptable when evaluated only on quantifiable data. Indirect or intangible costs seldom seem to get the same recognition as do nonmonetary benefits. Costs have been underestimated, say critics, because the local impact of major federal projects was not anticipated. Labor and material prices went up because of increased local demand caused by the project to the detriment of both project expenses and consumers in the community.

Projects are sometimes undertaken without the support of the residents in the area. Planners are thus forced into the position of telling residents what is good for them, which requires a selling campaign that may create costly delays or added community relations costs. Part of the problem could be alleviated by exposing a project to a public vote. However, difficulties of informing the voters about the issues involved and the remoteness of many projects from their base of funding (the taxpayers) limit the workability of decisions by vote. A choice between a football stadium and an art museum in a city would probably arouse enough interest by those affected to ensure a careful appraisal and representative turnout of voters. Yet an equal expenditure to develop a wildlife refuge might inspire less interest; even if all the affected voters could be given a chance to vote. On all questions of national objectives, there will likely be a few opponents, and many of these will be in the geographical locality where a project is to be carried out. Local protests can naturally be expected from people whose lives are disrupted by a project designed to serve the general welfare. It is the duty of policy makers to give consideration and adequate compensation to the local interests while supporting regional and national social interests.

Two benefit-cost analyses are described in the following pages. The applications are digested to point out certain highlights; the actual studies and

Methods for quantifying essentially intangible benefits are explored in Extension 10C.
techniques employed were much more exhaustive. Example 10.2 illustrates a feasibility study that disqualified public expenditures in solving a local problem. A postaudit to determine the actual benefit and cost of a program is featured in Example 10.3.

Example 10.2
Feasibility Investigation of a Water-Resource Project*

"The Corps of Engineers was directed by the Congress of the United States to make a study of Marys River Basin. The purpose of the study was to determine what could be done to reduce flood damages and to conserve, use, or develop the basin's water resource."

Six alternatives were investigated: (1) floodproofing individual structures, (2) building levees, (3) improving river channels, (4) instituting a system of land-owner-constructed dams, (5) implementing a system of small tributary reservoirs, and (6) erecting multipurpose reservoirs. The costs of flood-proofing, levees, and channel improvements were found to be much greater than the benefits, mainly because of the lack of secondary benefits. Landowner-constructed dams built with government financial support would have some localized benefits but would require an unreasonable amount of land for the storage obtained, and problems of balancing outflows made the alternative impractical.

The last two alternatives also proved to be economically infeasible, as shown in Figure 10.2. Flood-control benefits were based on historic flood damages updated to 1974 prices, estimates of flood-stage reductions by operation of the projects investigated, and projections of economic growth in the area. Recreation benefits were based on user-day projections. The cost of the cheapest alternative water supply was used in the analysis as a measure of the water-supply benefits that could reasonably be expected. Data provided by the U.S. Fish and Wildlife Service showed that fish and wildlife benefits resulting from any of the project alternatives would be negligible. Costs included construction and, as appropriate, land, relocation of roads, railroads, and utilities, fish-passage facilities, wildlife migration features, recreation developments, design costs, and construction supervision costs.

Example 10.3
Follow-up Study to Evaluate Benefit versus Cost for a Human-Resource Program†

The Roswell Employment Training Center was established in 1967 under contract with the Bureau of Indian Affairs. The Center was designed as a residential employment training program for Indian families, solo parents, and single adults. Emphasis is placed on training and counseling intended to aid trainees in adjusting to typical work situations and to living off the reservation.

Using earnings differentials to measure benefits from human-resource investment programs is a standard method of evaluation. The major resource costs are staff services, where salaries are taken as the measure of their value in an alternative use; other direct costs such as supplies; and the opportunity costs of training—the earnings forgone as a result of undertaking training.

In June 1970, a follow-up study was conducted by mail of all trainees who had entered Roswell Center between March 1968 and February 1970. The calculation of a benefit-cost ratio using the data from this survey is shown below. Of the group of 170 returning adequate survey forms, 70 were currently employed at an average hourly wage of $2.25 after training, whereas 82 members of the same group were employed before training was given. Because the average wage rate
<table>
<thead>
<tr>
<th>Costs, total</th>
<th>Noon Dam</th>
<th>Wren Dam</th>
<th>Tumutum Dam</th>
<th>Tributary Dam System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construction</td>
<td>$33,352,000</td>
<td>$52,944,000</td>
<td>$37,024,000</td>
<td>$19,090,000</td>
</tr>
<tr>
<td>Investment</td>
<td>57,386,000</td>
<td>99,165,000</td>
<td>41,174,000</td>
<td>23,313,000</td>
</tr>
<tr>
<td>Costs, annual interest and amortization</td>
<td>$3,380,000</td>
<td>$4,485,000</td>
<td>2,437,000</td>
<td>1,257,000</td>
</tr>
<tr>
<td>Operation, maintenance and replacement</td>
<td>210,000</td>
<td>180,000</td>
<td>170,000</td>
<td>135,000</td>
</tr>
<tr>
<td>Total average annual cost</td>
<td>$3,590,000</td>
<td>$3,665,000</td>
<td>$2,607,000</td>
<td>$1,392,000</td>
</tr>
</tbody>
</table>

Benefits, average annual

<table>
<thead>
<tr>
<th>Benefits</th>
<th>Noon Dam</th>
<th>Wren Dam</th>
<th>Tumutum Dam</th>
<th>Tributary Dam System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flood control</td>
<td>$945,400</td>
<td>$481,100</td>
<td>$758,500</td>
<td>$126,300</td>
</tr>
<tr>
<td>Recreation</td>
<td>573,000</td>
<td>345,000</td>
<td>401,000</td>
<td>163,000</td>
</tr>
<tr>
<td>Irrigation water supply</td>
<td>85,000</td>
<td>85,000</td>
<td>85,000</td>
<td>85,000</td>
</tr>
<tr>
<td>Municipal and industrial water supply</td>
<td>90,600</td>
<td>63,400</td>
<td>8,800</td>
<td>0</td>
</tr>
<tr>
<td>Fish and wildlife enhancement</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total average annual benefits</td>
<td>$1,694,000</td>
<td>$976,500</td>
<td>$753,300</td>
<td>$174,100</td>
</tr>
<tr>
<td>Benefit-cost ratio</td>
<td>0.47</td>
<td>0.27</td>
<td>0.29</td>
<td>0.27</td>
</tr>
</tbody>
</table>

*Construction cost plus interest during construction
The annual cost equivalent to the total cost spread over a 100-year period, computed at 7.8 percent interest rate

**FIGURE 10.2**
Benefit-cost analyses for alternative reservoir projects

After training was significantly higher than was the average wage before training, there was still a positive earnings differential associated with training, equal to $573 per year per trainee.

\[
\text{Additional annual earnings/trainee} = \frac{($4680 \times 70) - ($2808 \times 82)}{170} = $573
\]

Benefit of program. PW of extra annual earnings over 41 years (the number of remaining years in the labor force for trainees whose average age was 24) at a 6% discount rate:

\[
\text{Cost of program. Contracted cost for an average length of training of 8 months at $658 per month per trainee} = $5264
\]
\[
\text{Fargone earnings (number of trainees working before training times their monthly earnings times the months of training, divided by the total number of trainees)}
\]

\[
\text{Benefit of program. PW of extra annual earnings over 41 years (the number of remaining years in the labor force for trainees whose average age was 24) at a 6% discount rate:}
\]

\[
8674 = \frac{(82)(658)(41/2)}{1.06^{41}} - \frac{5264}{1.06^{41}} = $902
\]

\[
\text{Then, PW of total cost} = $5264 + $902 - 6166 = $8674
\]

\[
\frac{8674}{6166} = 1.41
\]

The benefit-cost analysis described above is the simplest possible for a human-resource investment program. Earnings before and after training are compared and differentials are projected, at a constant amount, over the years former trainees can be expected to remain in the labor force. The present value of these amounts are then compared with direct program costs plus the opportunity costs of trainees to estimate benefit-cost ratios. Every step of the analysis can
be challenged on theoretical and empirical grounds. Pretraining earnings may be understated; perhaps earnings differentials should be projected at growing, rather than constant amounts in future years; the lack of complete follow-up data may bias the estimated earnings differentials; a control group should actually be established; maybe 6 percent is not the "right" discount to use; an estimate of the value of the physical facilities in an alternative should actually be included as a cost—all this and probably more could be charged.

To focus on these niceties of analysis would be to miss an important aspect of program analysis. Analysis should not only ask the question, "Is this a good program?" or "Has the benefit-cost ratio been appropriately and accurately estimated and is it greater than unity?" but should provide program managers with the information they need to change particular aspects of their program in an effort to improve performance.*

Review Exercises and Discussions

**Exercise 1**

A proposal has been made to modify certain navigational aids that will decrease the cost of operation by $10,000 per year for the next 30 years. A loss in benefit during the same period amounts to $1000 per year. Conduct a benefit-cost evaluation with a discount rate of 10 percent, assuming that the modifications will be completed with little extra cost as part of the routine maintenance.

**Solution 1**

\[
\text{B/C} = \frac{-1000(P/A, 10, 30)}{-1000 \times (P/A, 10, 30)} = 0.1
\]

The criterion of B/C > 1.0 for acceptability is not applicable when both the benefit and cost of a project are negative. The ratio of savings to costs is obviously 10:1 in favor of the proposal. Thus, when both the numerator and denominator of a B/C ratio are negative, a ratio of less than 1.0 indicates acceptability. Moreover, a project with a positive B and a negative C is automatically accepted.

**Exercise 2**

The present worth of benefits and costs for two mutually exclusive "base" proposals are shown below, along with data for three supplementary projects which can be combined with either base to yield additional benefits. The supplementary projects are not mutually exclusive. Which combination of projects is preferred when resources are limited to $400,000?

<table>
<thead>
<tr>
<th>Basic Project Proposals</th>
<th>Supplementary Projects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Project</strong></td>
<td><strong>B</strong></td>
</tr>
<tr>
<td>P1</td>
<td>$300,000</td>
</tr>
<tr>
<td>P2</td>
<td>$450,000</td>
</tr>
<tr>
<td></td>
<td>$300,000</td>
</tr>
</tbody>
</table>


1 Adapted from G. A. Fleischer. *Benefit-Cost Analysis: An Introductory Exposition*. Tech Report 721, Department of Industrial and Systems Engineering, University of Southern California

62
Since P1 has a higher benefit-cost ratio than does P2, it is a logical base for combinations. Only combinations that approach but do not exceed the $400,000 limit are considered, because the objective is to maximize benefits with the resources available.

<table>
<thead>
<tr>
<th>Combination</th>
<th>Benefit</th>
<th>Cost</th>
<th>B - C</th>
<th>B/C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 + S1 + S2</td>
<td>$515,000</td>
<td>$300,000</td>
<td>$215,000</td>
<td>1.72</td>
</tr>
<tr>
<td>P1 + S3</td>
<td>600,000</td>
<td>300,000</td>
<td>300,000</td>
<td>2.0</td>
</tr>
<tr>
<td>P1 + S1 + S3</td>
<td>675,000</td>
<td>350,000</td>
<td>325,000</td>
<td>1.93</td>
</tr>
<tr>
<td>P1 + S2 + S3</td>
<td>740,000</td>
<td>400,000</td>
<td>340,000</td>
<td>1.85</td>
</tr>
</tbody>
</table>

Of these combinations, the one that spends all the available resources, P1 + S2 + S3, provides the greatest benefits ($740,000), yields the highest net benefits ($340,000), has a B/C ratio (1.85) greater than 1.0, and gives an incremental benefit-cost ratio exceeding unity.

\[
\text{ARIAC} = \frac{$740,000 - $675,000}{$400,000 - $350,000} = \frac{$65,000}{$50,000} = 1.3
\]

is the most satisfying.

However, the best possible combination is one that utilizes the other base, P2. The combination of P2 + S3 has a total benefit of $450,000 + $300,000 = $750,000 and a total cost of $250,000 + $150,000 = $400,000 to yield

\[
(B - C)_{P2 + S3} = $750,000 - $400,000 = $350,000
\]

and

\[
\frac{(B/C)_{P2 + S3}}{B/C_{P1 + S2 + S3}} = \frac{$750,000}{$400,000} = 1.875
\]

This combination might be overlooked by the preselection error of eliminating P2 from consideration because its benefit-cost ratio is less than the B/C for P1. It is also interesting to note that in this case the B/C ratio provides an accurate ranking of top contenders because both have the same total cost.

Benefit-cost ratios for two mutually exclusive alternatives have been calculated with disbenefits assigned to both numerator and denominator as shown.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Benefit (B)</th>
<th>Disbenefit (D)</th>
<th>Cost (C)</th>
<th>B/C = (B - D)/C</th>
<th>B/C = B/(C + D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>150</td>
<td>0</td>
<td>100</td>
<td>150 - 0(\times)100 = 1.5</td>
<td>150(100 + 0) = 1.5</td>
</tr>
<tr>
<td>A2</td>
<td>200</td>
<td>20</td>
<td>120</td>
<td>200 - 20(\times)120 = 1.5</td>
<td>200(120 + 20) = 1.43</td>
</tr>
</tbody>
</table>

What reply would you make to a claim that A1 is economically superior because its B/C ratio is equal to or better than that obtained from A2, regardless of the way disbenefits are handled?

The net benefit of A2 is 200 - 20 = 180 versus 150 - 100 = 50 for A1. If any preference is shown by the benefit-cost criterion, it is thus bestowed on A2. An incremental analysis conducted either way.

Solution 2

Exercise 3

Solution 3

63
<table>
<thead>
<tr>
<th>Project</th>
<th>B</th>
<th>C</th>
<th>ΔB/ΔC</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>150</td>
<td>100</td>
<td>1.5</td>
</tr>
<tr>
<td>A2</td>
<td>180</td>
<td>120</td>
<td>1.25</td>
</tr>
</tbody>
</table>

confirms that A2 is an acceptable project because its incremental B/C ratio is consistently greater than 1.0.

**Exercise 4**
Consider a freeway project that proposes a new four-lane highway leading from the downtown core of a city to a major arterial bypass in the suburbs. What factors besides the construction requirements could be considered in the total project cost?

**Solution 4**
Benefit-cost analyses for some major highway construction projects consume hundreds of pages. A few of the considerations include:

1. Air pollution from traffic flow
2. Expenditures for traffic control
3. Accident frequency and severity, past and forecast
4. Parking expectations in affected areas
5. Time and cost consequences of congestion, before and after
6. Shift of business competition
7. Accessibility of outlying jobs to central-city poor
8. Changes in property values due to accessibility
9. Reinvention of dwellings in the highway path, associated disruption of neighborhoods
10. Visual and auditory impact
11. Temporary effects of construction—wages, price of materials, living costs, employment, etc.
ACCELERATED COST RECOVERY SYSTEM (ACRS)

The Economic Recovery Act of 1981 had a pervasive impact on taxpayers, personal and corporate, big and small. Cuts in personal tax rates partially offset "bracket creep" caused by years of inflation. Estate tax provisions eased the tax burden on inheritances. Modernization of facilities and equipment was encouraged by larger investment tax credits and more lenient rules for research expenses. But the most sweeping revisions were in depreciation rules. The 1981 Act's intent was "to encourage economic growth through reduction of the tax rates for individual taxpayers, acceleration of capital cost recovery of investment in plant, equipment and real property, and incentives for savings, and other purposes."

Accelerated Cost Recovery System (ACRS) was the new method prescribed by the Economic Recovery Act of 1981 for the depreciation of all property placed in service after 1980.

ACRS Recovery Property

According to ACRS, recovery property is classified as either Section 1245 or Section 1250 class property. Section 1245 property generally includes all personal property, elevators and escalators, and special-purpose structures and storage facilities. Section 1250 property generally includes all real property.

The Internal Revenue Code of 1954, as amended, is the fundamental law governing federal taxes in the United States.

The ACRS acronym is informally voiced as "acres".

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that is subject to an allowance for depreciation and is not or never has been
depreciable personal property.

A recovery period is obtained by assigning an asset to a 3-, 5-, 10-, or 15-
year property class:

**3-year property.** Section 1245 class property with a present class life of 4 years
or less or used in connection with research and experimentation. Examples
include machinery and equipment used in research and development, automo-
biles and light trucks, and certain special tools.

**5-year property.** Section 1245 class property that is not 3-, 10-, or 15-year
public utility property. Examples include machinery and equipment not used
in research and development, office furniture, and fixtures. This class of
property appears frequently in engineering economic studies.

**10-year property.** Section 1250 class property with a present class life of 12.5
years or less. It includes certain public utility property. Other examples include
mobile homes and railroad cars.

**15-year real property.** Section 1250 class property that has a present class life
of more than 12.5 years. It includes all real property, such as buildings, other
than that designated as 5- or 10-year property. This class is separated into
low-income housing and all other real property, with each division having its
own schedule of ACRS percentages. (See Table 11.2.)

**ACRS Deductions**

The allowable ACRS deductions for tax purposes are simply calculated by
multiplying the unadjusted basis of property by the appropriate percentages.
Table 11.1 contains the recovery percentages for four classes of property.

Letting DC(n) be the depreciation charge in year n and p(n) be the
permissible deduction percentage for year n, we find that

\[
DC(n) = p(n) \times P
\]

where \( P \) is the unadjusted basis. Then the amount of the investment unrecovered
at the end of year \( n \) is

\[
BV(n) = P - \sum_{i=1}^{n} DC(i) - P \left(1 - \sum_{i=1}^{n} p(i)\right)
\]

where \( BV(n) \) is the book value, or unrecovered balance, at the end of year \( n \).

Unlike the 3-, 5-, or 10-year classes of property, and the 15-year public
utility property, the percentages for 15-year real property depend on when the
property is placed in service during the tax year. As shown in Table 11.2,
each month is represented by a column. The depreciation deduction for each
### TABLE 11.1

ACRS recovery percentages for property placed in service after 1985

<table>
<thead>
<tr>
<th>Year (n)</th>
<th>3-year Property</th>
<th>5-year Property</th>
<th>10-year Property</th>
<th>15-year Public Utility Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>33</td>
<td>20</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>43</td>
<td>32</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>24</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>16</td>
<td>14</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>12</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>10</td>
<td></td>
<td>9</td>
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<td>7</td>
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<td>8</td>
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<tr>
<td>10</td>
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<td>2</td>
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<td>5</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>0</td>
<td></td>
<td>4</td>
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<tr>
<td>12</td>
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<td>3</td>
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<tr>
<td>13</td>
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<td>14</td>
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<td>15</td>
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<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

### TABLE 11.2

ACRS recovery percentages for 15-year real property, other than low-income housing

<table>
<thead>
<tr>
<th>Month in Which 15-Year Real Property Is Placed in Service</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12%</td>
<td>11%</td>
<td>10%</td>
<td>9%</td>
<td>8%</td>
<td>7%</td>
<td>6%</td>
<td>5%</td>
<td>4%</td>
<td>3%</td>
<td>2%</td>
<td>1%</td>
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<tr>
<td>3</td>
<td>9</td>
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<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

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year of the recovery period is specified by the percentages in the column determined by the month in which the asset's use begins. Comparable percentages for low-income housing are given in IRS publication 534.

Many rulings surround the basic elements of ACRS presented in this brief coverage. The purpose here is to simply introduce concepts and communicate the major features, not to provide explanations detailed enough to construct depreciation accounts. Details are given in publications obtainable from IRS Forms Distribution Centers. The following three examples illustrate the general approach for calculating ACRS deductions under different investment conditions.

### Example 11.2

**ACRS 3-Year Property Depreciation and Book Values**

Three vehicles were placed in service in 1986. A car in January for $10,000, a light truck in May for $20,000, and a used automobile in December for $7000. What is the depreciation schedule for these vehicles? What would happen to the schedule if the vehicles were sold sometime in 1988?

**Solution 11.2**

All three purchases are 3-year recovery property and all were placed in service during the same year, 1986. Therefore, the total unadjusted basis for the year is $10,000 + $20,000 + $7000 = $37,000. The ACRS deductions over the recovery period are shown below.

If the three vehicles were sold any time in 1988, no deduction would be allowed for that year, but disposal after 1988 would have no effect on the deduction schedule. Note that the book value (undepreciated portion of the original purchase price) is $8140 for the year in which the sale is anticipated.

<table>
<thead>
<tr>
<th>Year</th>
<th>ACRS Deductions</th>
<th>Cumulative Depreciation</th>
<th>Book Value at End of Year</th>
<th>Year (n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1986</td>
<td>$37,000 × 0.33 = $12,210</td>
<td>$12,210</td>
<td>$24,790</td>
<td>1</td>
</tr>
<tr>
<td>1987</td>
<td>37,000 × 0.45 = 16,650</td>
<td>28,860</td>
<td>8,140</td>
<td>2</td>
</tr>
<tr>
<td>1988</td>
<td>37,000 × 0.22 = 8,140</td>
<td>37,000</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

The percentages applied to the $37,000 unadjusted basis in Example 11.2 are from the 3-year property column in Table 11.1. The month in which the 3-year property is put into service is irrelevant to the deduction percentage. Similarly, any prospective salvage value beyond the recovery period has no effect on the unadjusted basis or the percentages. Book values, representing the unrecovered investment, show that 67 percent of the investment remains unrecovered by the end of the first year—$12,210. The vehicles are fully depreciated by the end of 3 years, which produces a book value of zero for 1989, assuming that the vehicles are still in service at that date.
Example 11.3
ACRS 5-Year Property Purchased with a Trade-In
An old typewriter that has an adjusted basis of $125 is traded in on a new model that has a fair market value of $925. The new typewriter was purchased in December 1980 by a cash payment of $800, but the machine was not delivered until January 1987. What is the ACRS deduction for 1987?

Solution 11.3
Since the typewriter was not put into service until 1987, the first deduction takes place in that year. Typewriters are classified as 5-year property, so the first year percentage is 20 percent, as indicated in Table 11.1. The unadjusted basis of the new machine is the sum of the cash outlay ($800) plus the value of the trade-in as specified by its adjusted basis ($125): $925. Therefore, the ACRS deduction for 1987 $925 \times 0.20 = \$185$.

Example 11.4
ACRS 15-Year Real Property Depreciation and Resale
An engineering consulting firm uses the calendar year as its tax year. The firm purchased an adjacent building in March 1986 for $230,000, not including land, and converted it into office space at a cost of $50,000. Renovations were completed by September and personnel from the firm moved in during that month. What are the permissible deductions for the first 3 years of occupancy? What deduction would be allowed 1989 if the building was resold for $300,000 on May 1, 1989?

Solution 11.4
An office building is classified as 15-year real property. Deductions in this class are indexed to the month the property is placed in service. The consulting firm’s building was placed in service during the ninth month, at which time its unadjusted basis was $230,000 + $50,000 = $280,000. The percentage for the first year is then 4 percent, allowing a 1986 deduction of $280,000 \times 0.04 = \$11,200$. The same column, month 9 in Table 11.2, indicates percentages of 11 and 10 percent for years 2 and 3, respectively, that produce deductions of $30,800 in 1987 and $28,000 in 1988.

If the building were sold on May 1, 1989, the allowable deduction for 1989 would be 4/12 (4 months of the year have passed) of the percentage designated for the fourth year of service: $280,000 \times 0.09 \times 4/12 = \$8400$. The price obtained from the sale has no effect on the deduction, but must be included in income tax calculations.

ALTERNATIVE ACRS METHOD

In lieu of using the percentages given in Tables 11.1 and 11.2, deductions may be calculated using a straight-line adaptation of the ACRS method. When this alternative method is utilized, the unadjusted basis of the recovery property is divided by the number of years in a recovery period selected from the options given in Table 11.3.

Straight-line percentages for $p(n)$:

\[
\begin{align*}
N = 3: & \quad 0.33333 \\
N = 5: & \quad 0.20000 \\
N = 10: & \quad 0.10000 \\
N = 12: & \quad 0.08333 \\
N = 15: & \quad 0.06667 \\
N = 25: & \quad 0.04000 \\
N = 35: & \quad 0.02857 \\
N = 45: & \quad 0.02222
\end{align*}
\]

Straight-line deductions are calculated from the fraction $p(n)$, where $N$ is the elected recovery period. A percentage is applied to the unadjusted basis as:

\[
\text{Full-year deduction} = p(n) \times \text{unadjusted basis} \times \frac{1}{N}
\]
TABLE 11.3
Permissible recovery periods for the alternative ACRS method that uses a straight-line approach to calculate annual deductions

<table>
<thead>
<tr>
<th>ACRS Recovery Property Classification</th>
<th>Optional Recovery Periods (N) for Deductions Based on the Alternative Straight-Line Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-year</td>
<td>3, 5, or 12 years</td>
</tr>
<tr>
<td>5-year</td>
<td>5, 12, or 25 years</td>
</tr>
<tr>
<td>10-year</td>
<td>10, 25, or 35 years</td>
</tr>
<tr>
<td>15-year real</td>
<td>15, 35, or 45 years</td>
</tr>
<tr>
<td>15-year public utility</td>
<td>15, 35, or 45 years</td>
</tr>
</tbody>
</table>

where \( p(n) \) is constant over the recovery period. No salvage value is allowed in the calculation. Once a value of \( N \) is elected, the same value must be applied to all property in a 3-, 5-, or 10-year class placed in service during the same year; this percentage cannot be changed during the recovery period. However, different percentages may be elected for a property class in subsequent years.

Only a half year of depreciation can be claimed the first year that the property is placed in service. A full year’s deduction can be taken for the rest of the recovery period, and then another half-year’s depreciation is applied in the year following the recovery period. If the property is sold during the recovery period, no deduction is allowed in the year of the sale.

Example 11.5
ACRS Straight-Line Deductions with the Half-Year Convention

A new machine is placed in service at a research lab in December 1986. The purchase price is $10,000. The company elects to use a 5-year recovery period even though the machine is classified as 3-year recovery property. What is the ACRS deduction schedule for the machine if it is kept in use for 6 years?

Solution 11.5
The 5-year straight-line percentage is \( 100\% / 5 = 20\% \). Regardless of the month an asset is placed in service (other than 15-year property), the half-year convention applies:

\[
P = \frac{1}{N} \times \frac{1}{2}
\]

Thereafter, the full deduction is registered for the remaining years of the elected recovery period—in this case, 4 years. Then the final half-year deduction is taken in the sixth year, as shown below.

<table>
<thead>
<tr>
<th>Year</th>
<th>ACRS Straight-Line Deduction</th>
<th>Cumulative Depreciation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1986</td>
<td>$10,000 \times 0.20 \times \frac{1}{2} = $1000</td>
<td>$1,000</td>
</tr>
<tr>
<td>1987 through 1991</td>
<td>$10,000 \times 0.20 = $2000</td>
<td>$9,000</td>
</tr>
<tr>
<td>1992</td>
<td>$10,000 \times 0.20 \times \frac{1}{2} = $1000</td>
<td>$10,000</td>
</tr>
</tbody>
</table>

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OTHER DEPRECIATION METHODS

Before ACRS was enacted, other methods were used to figure depreciation. Assets placed in service prior to 1981 and property that does not qualify for ACRS must still use these methods, but they cannot be applied to property that now qualifies for ACRS. Whereas many different methods of figuring depreciation are acceptable, it is the taxpayer's responsibility to show that a method is "reasonable," if it is questioned.

Three of the most prominent depreciation methods are examined in the following pages: straight-line, sum-of-digits, and declining balance. They require input information about an asset's basis, useful life, and salvage value expected at the end of its useful life. Different annual depreciation charges result from the application of each method, and all three differ from ACRS. These methods are illustrated by reference to the problem data given in the table for Examples 11.6 to 11.8.

The symbols used in the development of the formulas are

\[ P = \text{purchase price (unadjusted basis) of asset} \]
\[ S = \text{salvage value or future value at end of asset's useful life} \]
\[ N = \text{useful life of asset} \]
\[ n = \text{number of years of depreciation or use from time of purchase} \]
\[ DC = \text{annual charge for depreciation} \]
\[ BV = \text{book value shown on accounting records (adjusted basis)} \]

Straight-Line Method

Straight-line depreciation is the simplest method to apply and the most widely used of the depreciation methods. The annual depreciation is constant. The book value is the difference between the purchase price and the product of the number of years of use times the annual depreciation charge:

\[ DC = \frac{P - S}{N} \]

\[ BV_{\text{end of year } n} = P - \frac{n}{N}(P - S) \]

Data and problem statement for Examples 11.6 through 11.8

Small computers were purchased by a public utility at a cost of $7000 each. Past records indicate that they should have a useful life of 5 years after which they can be sold for an average price of $1000. The company currently has a cost of capital of 7 percent. Determine

a. The depreciation charge during year 1
b. The depreciation charge during year 2
c. The depreciation reserve accumulated by the end of year 3
d. The book value of the computers at the end of year 3

To be acceptable to IRS, a depreciation method must be used consistently and depreciate property no faster than the double-rate-declining-balance method.

The basis for depreciation is the same as that used for figuring the gain on a sale. The original basis is usually the purchase price.
Example 11.6
Straight-Line Depreciation Applied to the Basic Data

Solution 11.6

a and b Since the annual depreciation cost is constant, the charges for both the first and second year are
d

\[ DC = \frac{P - S}{N} = \frac{7000 - 1000}{5} = \$1200/\text{year} \]

or, with the book value considered as the difference between the purchase price and the amount accumulated in the depreciation reserve.

\[ BV(3) = 7000 - 3(1200) = \$3400 \]

The depreciation reserve at the end of the third year is the sum of the annual depreciation charges for the first 3 years and is equal to \( \frac{3 \times \$1200}{\$3600} \).

Sum-of-Digits Method

This method can be applied only to property that meets requirements for double-declining-balance depreciation.

The sum-of-digits method provides a larger depreciation charge during the early years of ownership than it does during the later years. The name is taken from the calculation procedure. The annual charge is the ratio of the digits representing the remaining years of life \( (N - n + 1) \) to the sum of the digits for the entire life \( 1 + 2 + 3 + \cdots + N \) multiplied by the initial price minus the salvage value \( (P - S) \). Thus, the annual charge decreases each year from a maximum the first year:

\[ DC = \frac{N - n + 1}{1 + 2 + 3 + \cdots + N} (P - S) \]

\[ = \frac{2(N - n + 1)}{N(N + 1)} (P - S) \]

\[ BV(n) = \frac{2(1 + 2 + \cdots + (N - n))}{N(N + 1)} (P - S) \times S \]

Example 11.7
Sum-of-Digits Depreciation Applied to the Basic Data

Solution 11.7

a The sum of digits for the 3-year useful life is

\[ 1 + 2 + 3 + 4 + 5 = 15 \]

or

\[ \frac{N(N + 1)}{2} = \frac{5(5 + 1)}{2} = 15 \]

which is the denominator of the formula for

\[ DC(1) = \frac{N - n + 1}{15} (P - S) \]

\[ = \frac{5 - 1 + 1}{15} (7000 - 1000) \]

\[ = \frac{5}{15} \times 6000 = \$2000 \]
b. After the first year, only 7 years remain in the useful life. Therefore, with \( N = n + 1 = 5 \), \( 2 + 1 = 4 \),

\[
DC(2) = \frac{4}{15} \times 6000 = 1600
\]

c. The ratio for calculating the depreciation reserve has a numerator equal to the sum of digits representing the years during which the reserve was built up:

\[
\text{Depreciation reserve at end of year 3} = \frac{5 + 4 + 3}{15} \times 6000 = 4800
\]

d. \( BV(3) = P - \text{depreciation reserve at end of year 3} \)

\[
= 7000 - 4800 = 2200
\]

or, by formula,

\[
BV(3) = \frac{2(1 + 2 + \cdots + (N - n))}{N(N + 1)} (P - S) + S
\]

\[
= \frac{2(1 + (5 - 3))}{5(5 + 1)} \times 6000 + 1000
\]

\[
= \frac{6}{30} \times 6000 + 1000 = 1200 + 1000 = 2200
\]

---

**Declining-Balance Method**

The declining-balance method is another means of amortizing an asset at an accelerated rate early in its life, with corresponding lower annual charges near the end of service. An important point with this method is that salvage value must be greater than zero. A depreciation rate is calculated from the expression

\[
\text{Depreciation rate} = 1 - \left(\frac{S}{P}\right)^{1/N}
\]

which requires a positive value for \( S \) in order to be feasible. This constant rate is applied to the book value for each depreciation period. Since the undepreciated balance decreases each year, the depreciation charge also decreases, and

\[
BV(n) = P \left(1 - \text{depreciation rate}\right)^n
\]

\[
= P \left(1 - \left(1 - \left(\frac{S}{P}\right)^{1/N}\right)\right)^n
\]

\[
= P \left(\frac{S}{P}\right)^n
\]

\[
DC(n) = BV(n) \left(1 - \frac{S}{\sqrt[2]{P}}\right)
\]

A much more widely used version of the declining-balance method, allowed by the income-tax code since 1954, is based on a depreciation rate which does not depend on the \( S/P \) ratio. Under certain circumstances a rate is allowed that is twice as great as would be proper under the straight-line method. Under other circumstances, the rate is limited to 1.5 or 1.25 times that of the straight-line method.

To qualify for depreciation by the double declining-balance method, property must be:
- Tangible and personal
- \( N > 3 \) years
- Placed in service before 1981
- Acquired new or built after 1953

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When the maximum rate is used,

\[ \text{Depreciation rate}_{\text{max}} = \frac{200\%}{N} \]

It is called the double-declining-balance method of depreciation. It has the same characteristics as does the declining-balance method.

---

**Example 11.8**

Double-Declining-Balance Depreciation Applied to the Basic Data in Table 11.4

**Solution 11.8**

\( a \) Given that

\[ \text{Depreciation rate}_{\text{max}} = \frac{200\%}{5} = 40\% \text{ or } 0.4 \]

\[ \text{DC}(1) = P(0.4) = \$7000(0.4) = \$2800 \]

\( b \) \[ \text{DC}(2) - \text{BV}(1)(0.4) \]

\[ = (\$7000 - \$2800)(0.4) \]

\[ = \$4200(0.4) = \$1680 \]

\( c \) The depreciation reserve at the end of year 3 is the sum

\[ \text{DC}(1) + \text{DC}(2) + \text{BV}(2)(0.4) \]

\[ = \$2800 + \$1680 + \$2520(0.4) \]

\[ = \$4480 + \$1008 = \$5488 \]

\( d \) \[ \text{BV}(3) = P - \text{depreciation reserve} \]

\[ = \$7000 - \$5488 \]

\[ = \$1512 \]

or \[ \text{BV}(3) = P(1 - \text{rate})^3 \]

\[ = \$7000(0.6)^3 \]

\[ = \$1512 \]

---

A difficulty may arise with the use of double-declining-balance depreciation because the salvage value is not included in the calculation of depreciation charges. Continuing Example 11.8 to determine the book value at the end of year 5, we find that

\[ \text{BV}(5) = P(1 - \text{depreciation rate})^5 - \$7000(0.6)^5 = \$544 \]

which is well below the anticipated salvage value of \$1000.

Since the IRS does not permit depreciation charges that drop the book value below the salvage value, it is necessary to halt depreciation when \( \text{BV} = S \). Referring again to Example 11.8, we find that

\[ \text{DC}(4) = \$1512(0.4) = \$605 \]

causing \( \text{BV}(4) = \$1512 - \$605 = \$907 \), which is less than \( S = \$1000 \). Therefore, the depreciation charge in year 4 is limited to \( \$1512 - \$1000 = \$512 \), and no depreciation charge is made in year 5. This pattern of double-declining-balance depreciation is shown in Figure 11.1.

It is not uncommon for the book value calculated by double-declining-
balance depreciation to exceed the asset's value at the end of its life. This situation always occurs when \( S = 0 \). Then it is usually advantageous to switch to straight-line depreciation. Since the IRS allows a switch in any year, the preferred time is the one which provides a present-worth tax advantage by deferring taxes to later years (see Table 11.4). The time to switch from double-declining-balance to straight-line depreciation is when the straight-line depreciation charge on the undepreciated portion of the asset's value exceeds the double-declining-balance allowance. The undepreciated portion is the difference between the asset's book value in a given year and its salvage value: \( DC = (BV(n) - S)/(N - n) \). The procedure is demonstrated in Example 11.9.

**Example 11.9**

Switch from Double-Declining-Balance to Straight-Line Depreciation

An asset has a first cost of $7000, a 5-year useful life, and no salvage value. Determine an accelerated depreciation schedule in which \( BV(N) = 0 \).
### TABLE 11.4

Depreciation pattern for a switch from the double-declining-balance (DDB) method to the straight-line (SL) method. The switch takes place at the end of year 3 when the SL depreciation charge exceeds the DDB charge.

<table>
<thead>
<tr>
<th>End of Year</th>
<th>DDB Depreciation Charges</th>
<th>Book Value with DDB</th>
<th>SL Depreciation on the Undepreciated Balance</th>
<th>Book Value, DDB→SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,500$</td>
<td>7000</td>
<td>$1,500$</td>
<td>$2,000$</td>
</tr>
<tr>
<td>1</td>
<td>$1,500$</td>
<td>4700</td>
<td>$1,500$</td>
<td>$2,000$</td>
</tr>
<tr>
<td>2</td>
<td>$1,500$</td>
<td>2520</td>
<td>$1,500$</td>
<td>$2,000$</td>
</tr>
<tr>
<td>3</td>
<td>$1,500$</td>
<td>1510</td>
<td>$1,500$</td>
<td>$2,000$</td>
</tr>
<tr>
<td>4</td>
<td>$605$</td>
<td>907</td>
<td>$793$</td>
<td>$2,000$</td>
</tr>
<tr>
<td>5</td>
<td>$363$</td>
<td>544</td>
<td>$363$</td>
<td>$2,000$</td>
</tr>
</tbody>
</table>

### Solution 11.9

Applying the double-declining-balance (DDB) method, as was done for the same $P$ and $N$ values in Example 11.8, we know that

$$BV(5) = 7000(0.6)^5 = 5544$$

which is higher than the zero salvage value. Therefore, a switch to the straight-line (SL) method is advisable to get the ending book value down to zero. The pattern of depreciation charges and book values resulting from the DDB method and the composite method of starting with DDB and switching to SL (DDB→SL) is shown in Table 11.4.

At the end of year 2 the book value resulting from DDB depreciation is $2520$, which equals the undepreciated balance because $S = 0$. Then the SL charges for the last 3 years would be

$$DC_{SL} = \frac{2520 - 0}{3} = 840$$

Since this annual charge is less than the DDB charge for year 3 ($1008$), accelerated depreciation is continued another year. Then $BV(3) = 1510$ and the SL depreciation charge for each of the last 2 years is $1510/2 = 755$. This is larger than the DDB depreciation charge for year 4 ($605$) and signals the time to switch.